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# **Watershed Sciences 6900**

## **FLUVIAL HYDRAULICS & ECOHYDRAULICS**

# **GRADUALLY VARIED FLOW & WATER-SURFACE PROFILES**

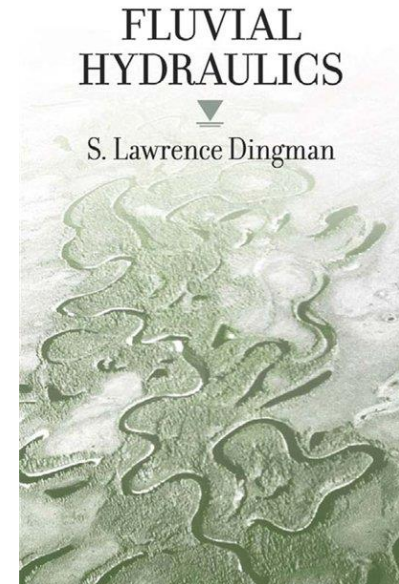
Joe Wheaton



# FOLLOW UP READING FOR TODAY

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- Chapters 9.1 -9.4



# RECALL STEADY GRADUALLY VARIED FLOW

$$Z_U + Y_U \cdot \cos\theta_0 + \frac{\alpha_U \cdot U_U^2}{2 \cdot g} = Z_D + Y_D \cdot \cos\theta_0 + \frac{\alpha_D \cdot U_D^2}{2 \cdot g} + \Delta H \quad \text{EQ 8.8b}$$

- Depth can vary!
- Assumed that pressure distribution was hydrostatic (i.e. streamlines no significantly curved)
- We call this **steady gradually varied flow**

## 1. Steady Uniform Flow

$$a_G - (a_V + a_T) = 0 \quad \text{EQ 7.2}$$

## 2. Steady Nonuniform Flow

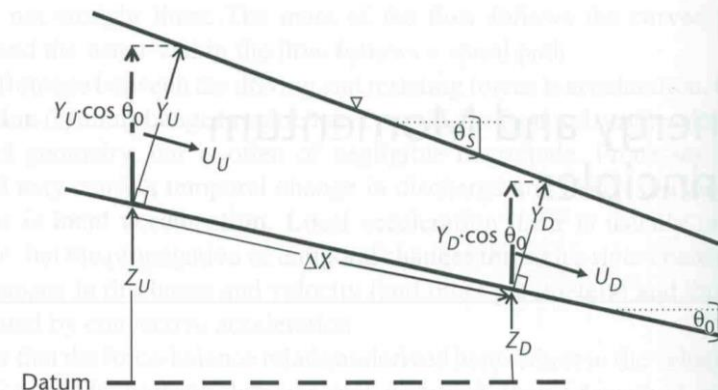
$$(a_G + a_p) - (a_V + a_T + a_X) = 0$$

$$(a_G + a_p) - (a_V + a_T) = a_X \quad \text{EQ 7.3}$$

## 3. Unsteady Nonuniform Flow

$$(a_G + a_p) - (a_V + a_T + a_X + a_t) = 0$$

$$(a_G + a_p) - (a_V + a_T) = (a_X + a_t) \quad \text{EQ 7.4}$$



# TOTAL ENERGY $h$

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- The total energy  $h$  of an element is the sum of its potential energy  $h_{PE}$  & its kinetic energy  $h_{KE}$ :

$$h = h_{PE} + h_{KE} \quad \text{EQ 8.1}$$

- $h_{PE}$  is the sum of gravitational potential energy  $h_G$  and pressure potential energy  $h_P$ :

$$h_{PE} = h_G + h_P$$

- $\therefore$  total energy  $h$  is:

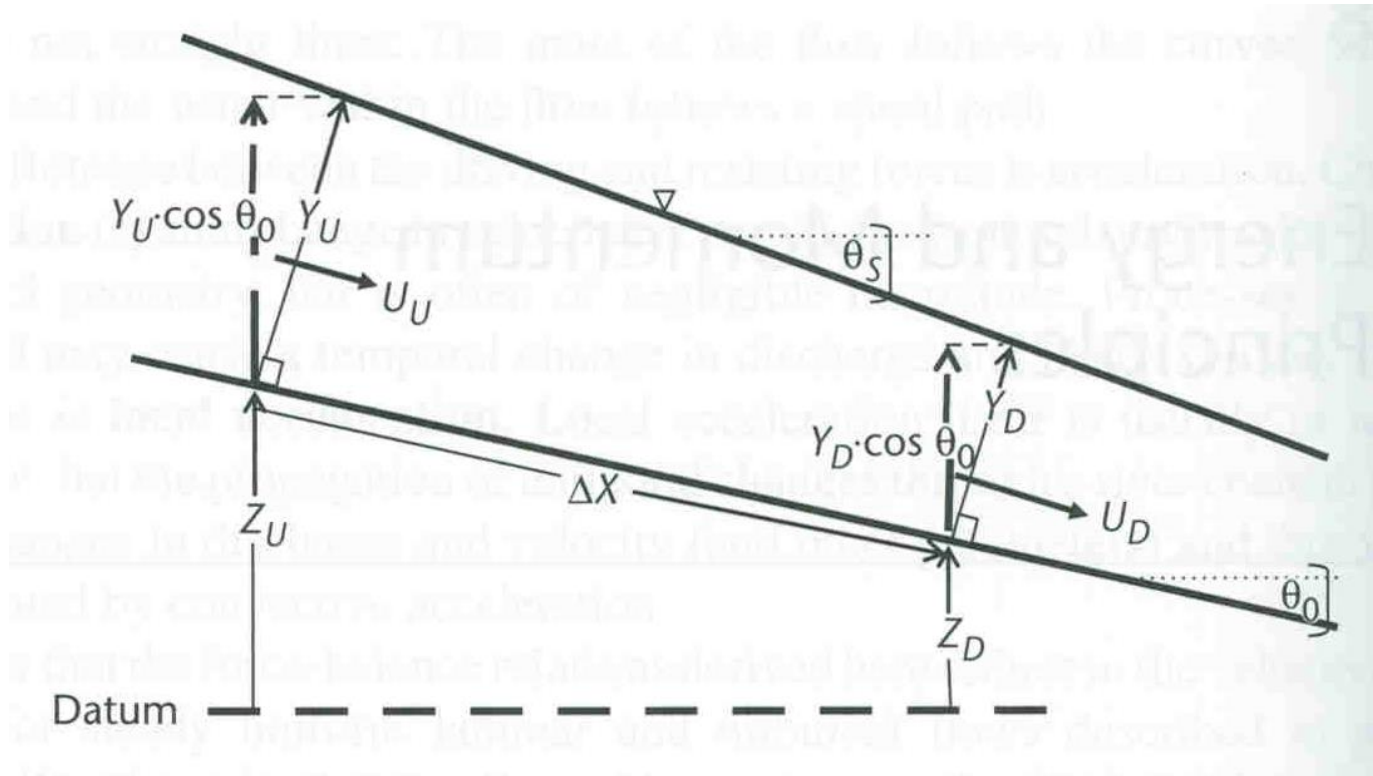
$$h = h_G + h_P + h_{KE}$$

EQ 8.2

Energy quantities are expressed as energy [F L] divided by weight [F], which is called head [L]



# DEFINITION DIAGRAM FOR ENERGY IN 1D



- Used for derivation of the macroscopic one-dimensional energy equation
- Flow through reach is steady



# TOTAL MECHANICAL ENERGY AT A CROSS SECTION

- Using the channel bottom as a reference point, at cross section  $i$  we can write the integrated **gravitational head** (AKA elevation head),  $H_{Gi}$ :

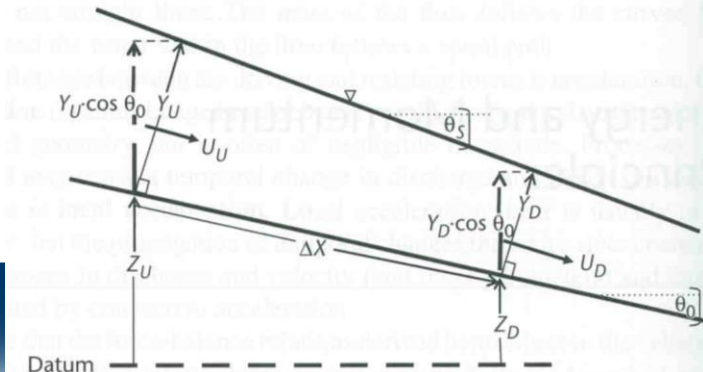
$$H_{Gi} = Z_i \quad \text{EQ 8.3}$$

- Where  $Z_i$  is the elevation of the channel bottom, and the integrated **pressure head**  $H_{Pi}$  is:

$$H_{Pi} = Y_i \cdot \cos\theta_0 \quad \text{EQ 8.4}$$

- Kinetic energy head  $H_{KE}$  for a fluid element with velocity  $u$  is given by:

$$h_{KE} = \frac{u^2}{2 \cdot g} \quad \text{EQ 8.5}$$



# VELOCITY COEFFICIENT $\alpha$ FOR ENERGY

- Because velocity varies from point to point in a cross section, we need a fudge factor  $\alpha_i$  to account for this variation, so the **velocity head**  $H_{KEi}$  is:

$$h_{KEi} = \frac{\alpha_i \cdot U_i^2}{2 \cdot g} \quad \text{EQ 8.6}$$

## BOX 8.1 Velocity Coefficients for Energy and Momentum: Definitions

In macroscopic one-dimensional formulations of the energy (section 8.1) and momentum (section 8.2) relations, the velocity  $U_i$  is the velocity averaged over cross section  $i$ . This is the velocity that we use to compute the kinetic energy flux and the momentum flux through each section. However, because in general velocity  $u$  varies from point to point in each section, **coefficients are required to compute the true kinetic energy** and momentum fluxes using the average velocity. Here, we derive the general expressions for these coefficients for cross sections of arbitrary shape and

(Continued)

## BOX 8.1 Continued

velocity distribution. Box 8.2 describes approaches to estimating  $\alpha$  and  $\beta$  and computes their values for the case of a wide rectangular channel and the Prandtl-von Kármán (P-vK) velocity distribution.

### Discharge

Referring to figure 8.2a, the elemental discharge,  $dQ$ , through an elemental area,  $dA$ , is

$$dQ = u \cdot dA, \quad (8B1.1)$$

where  $u$  is the elemental velocity. The total discharge,  $Q$ , is

$$Q = \int_A dQ = \int_A u \cdot dA = U \cdot A, \quad (8B1.2)$$

where  $A$  is the flow cross-sectional area.

### Energy Coefficient, $\alpha$

Referring to figure 8.2a, the weight of water passing through  $dA$  per unit time with velocity  $u$  is  $\gamma \cdot u \cdot dA$ , where  $\gamma$  is weight density. From equation 4.39, the kinetic energy passing through the element per unit time equals

$$\frac{u^2}{2 \cdot g} \cdot \gamma \cdot u \cdot dA = \frac{\gamma}{2 \cdot g} \cdot u^3 \cdot dA. \quad (8B1.3)$$

The total flow rate of kinetic energy through the cross section is found by integrating (8B1.3):

$$\int_A \frac{\gamma}{2 \cdot g} \cdot u^3 \cdot dA = \left( \frac{\gamma}{2 \cdot g} \right) \cdot \int_A u^3 \cdot dA. \quad (8B1.4)$$

If we simply use the average velocity  $U$  to compute the flow rate of kinetic energy through a section, we get

$$\gamma \cdot Q \cdot \left( \frac{U^2}{2 \cdot g} \right) = \gamma \cdot A \cdot \left( \frac{U^3}{2 \cdot g} \right). \quad (8B1.5)$$

The **energy coefficient**,  $\alpha$ , is defined as the ratio of the true kinetic-energy flow rate (equation 8B1.4) to the flow rate computed using the average velocity (equation 8B1.5):

$$\alpha \equiv \frac{\left( \frac{\gamma}{2 \cdot g} \right) \cdot \int_A u^3 \cdot dA}{\left( \frac{\gamma}{2 \cdot g} \right) \cdot U^3 \cdot A} = \frac{\left( \frac{1}{A} \right) \cdot \int_A u^3 \cdot dA}{U^3}. \quad (8B1.6)$$

That is, it is the ratio

$$\alpha \equiv \frac{\text{average of cubed velocities}}{\text{cube of average velocity}}. \quad (8B1.6)$$

# A CONVENIENT OUTCOME OF BOX 8.2

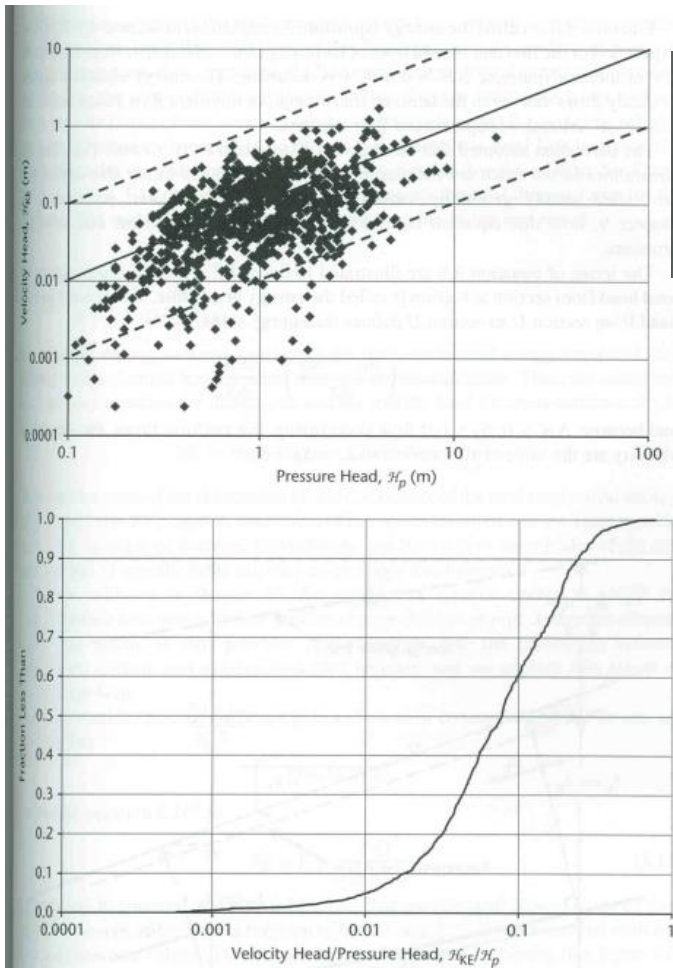


Figure 8.4 compares velocity heads and pressure heads for a database of measurements on 931 reaches.<sup>1</sup> A value of  $\alpha = 1.3$  is assumed in calculating velocity head. Figure 8.4 reveals that, typically, velocity head is less than 10% of pressure head. Because velocity head is often relatively small, determining the exact value of  $\alpha$  is not usually a critical concern.

Velocity Head  $H_{KE}$  tends to be at least an order of magnitude smaller than pressure head  $H_P$

$\alpha \approx 1.3$  good enough typically...

Figure 8.4 (a) Scatter plot of velocity head,  $H_{KE}$ , versus pressure head,  $H_P$ , for 931 flows in natural channels. The upper dashed line represents  $H_{KE} = H_P$ ; the solid line,  $H_{KE} = 0.1 \cdot H_P$ ; and the lower dashed line,  $H_{KE} = 0.01 \cdot H_P$ . (b) Cumulative-frequency diagram for the ratio  $H_{KE}/H_P$  for the flows plotted in (a). These data show that  $H_{KE}$  is almost always less than  $0.5 H_P$  and is commonly less than  $0.1 \cdot H_P$ .



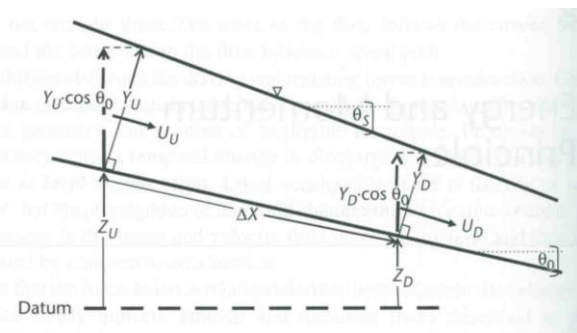
# TOTAL HEAD $H_i$ AT A CROSS SECTION $i$

- 'The total mechanical energy-per-weight, or **total head**  $H_i$ , at cross-section  $i$ , is the sum of the gravitational, pressure and velocity heads:

$$h = h_G + h_P + h_{KE} \quad \longrightarrow \quad H_i = H_{Gi} + H_{Pi} + H_{KEi}$$

$$H_i = Z_i + Y_i \cdot \cos\theta_0 + \frac{\alpha_i \cdot U_i^2}{2 \cdot g}$$

EQ 8.6



But, this is just at one cross section  $i$ ... What about for this diagram?

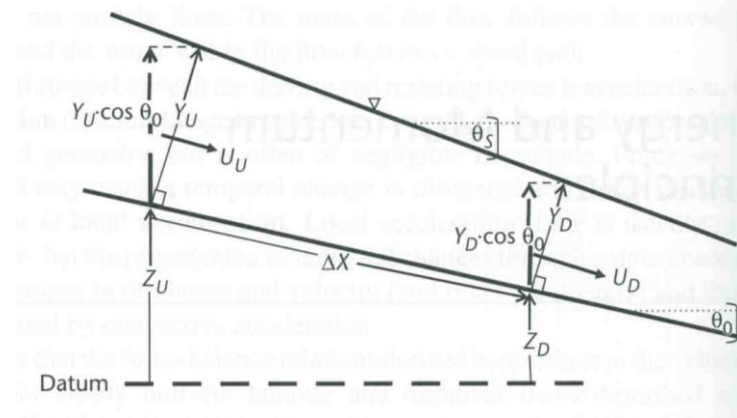


# APPLY $H_i$ BETWEEN TWO CROSS SECTIONS

$$H_i = Z_i + Y_i \cdot \cos\theta_0 + \frac{\alpha_i \cdot U_i^2}{2 \cdot g}$$

EQ 8.6

- What is the change in in cross-sectional integrated energy from an upstream section ( $i = U$ ) to a downstream section ( $i = D$ )?



$$H_{GU} + H_{PU} + H_{KEU} = H_{GD} + H_{PD} + H_{KED} + \Delta H$$

EQ 8.8a

$$Z_U + Y_U \cdot \cos\theta_0 + \frac{\alpha_U \cdot U_U^2}{2 \cdot g} = Z_D + Y_D \cdot \cos\theta_0 + \frac{\alpha_D \cdot U_D^2}{2 \cdot g} + \Delta H$$

EQ 8.8b

- Where  $\Delta H$  is the energy lost (converted to head) per weigh of fluid, or **head loss**.

**This is the ENERGY EQUATION!**



# THE ENERGY EQUATION APPLIES WHERE?

$$Z_U + Y_U \cdot \cos\theta_0 + \frac{\alpha_U \cdot U_U^2}{2 \cdot g} = Z_D + Y_D \cdot \cos\theta_0 + \frac{\alpha_D \cdot U_D^2}{2 \cdot g} + \Delta H \quad \text{EQ 8.8b}$$

- Depth can vary!
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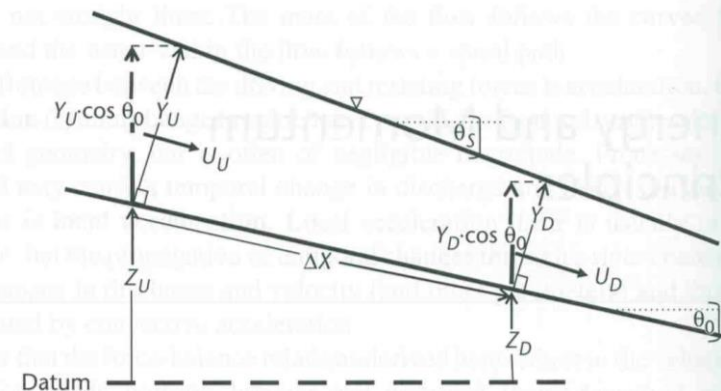
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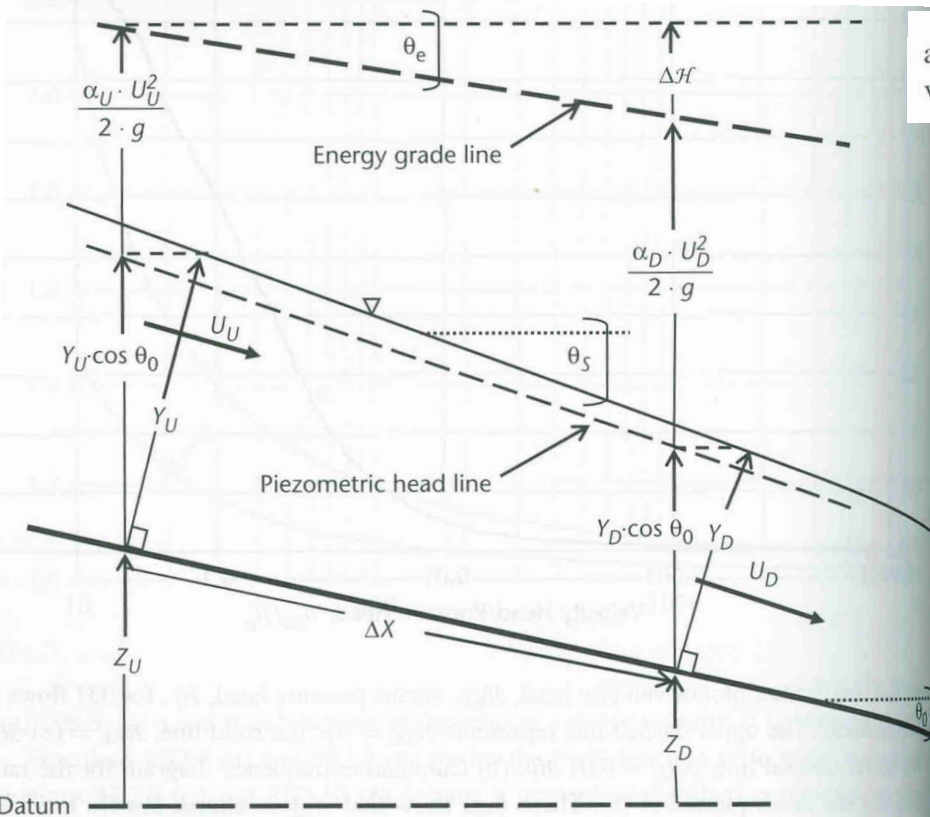


# SOME IMPORTANT IMAGINARY LINES

- The line representing the total head from section to section is called the **energy grade line**
- The slope  $S_E$  of the energy grade line is the **energy slope**:

$$S_E \equiv - \left( \frac{H_D - H_U}{\Delta X} \right) = \frac{\Delta H}{\Delta X} = \quad \text{EQ 8.9}$$

and because  $\Delta H > 0$ ,  $S_e > 0$  if flow is occurring. For uniform flows, the depth and velocity are the same at all sections, so  $S_e = S_s = S_0$ .



- The line representing the total potential energy from section to section is called the **piezometric head line**

the **piezometric head line**. The slope of this line represents the gradient of potential energy that induces flow; therefore, the line must always slope downstream. Because  $\cos \theta_0 < 1$ , the piezometric head line lies some distance below the water surface. However, the slopes of most streams are almost always less than 0.1, so  $\cos \theta_0 > 0.995$  and can be taken to be equal to 1; that is, the piezometric head line is essentially coincident with the water surface and has a slope equal to the surface slope  $S_s$ . Recall from section 7.3.1.3 that the surface slope represents the total driving force for the flow (i.e., the sum of the gravitational and pressure forces per unit mass).

# TODAY'S PLAN

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## GRADUALLY VARIED FLOW & WATER SURFACE PROFILES (Applied to 1D flows)

- I. Water Surface Profile Classification (§ 9.2)**
  - Controls (§ 9.3)
- II. Computation of WS Profiles (§ 9.4)
  - I. Theoretical Basis
  - II. Standard-Step Approach
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# FLOW PROFILES CLASSIFIED

- **Normal Depth**  $Y_n =$

$$\left( \frac{n_M \cdot Q}{u_M \cdot W \cdot S_0^{1/2}} \right)^{3/5} :$$

- **Critical Depth**  $Y_c \equiv$

$$\left( \frac{Q}{g^{1/2} \cdot W} \right)^{2/3}$$

- **Mild Slope:** Uniform flow subcritical &  $Y_n > Y_c$

- **Steep Slope:** Uniform flow supercritical &  $Y_n < Y_c$

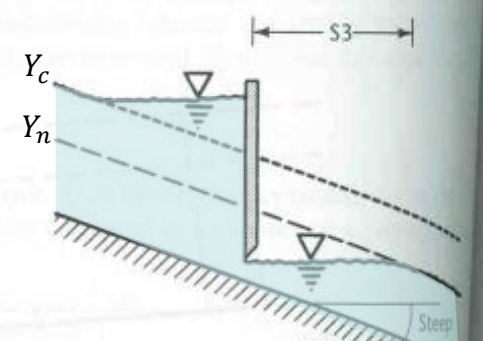
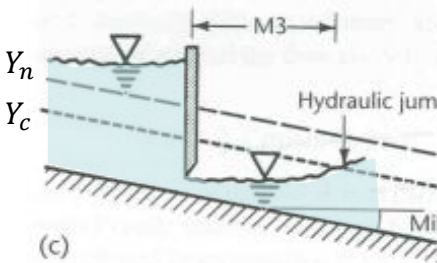
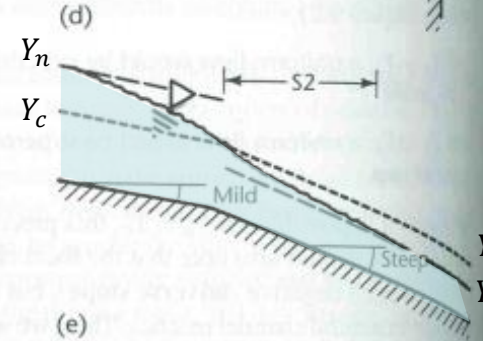
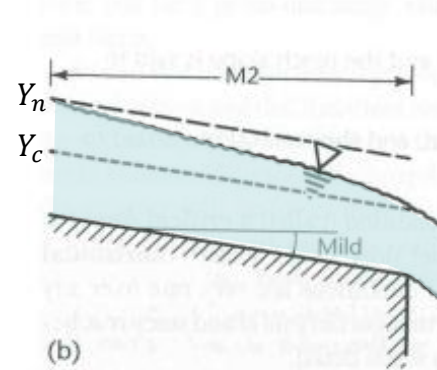
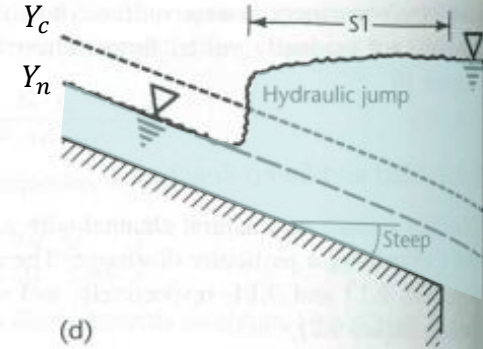
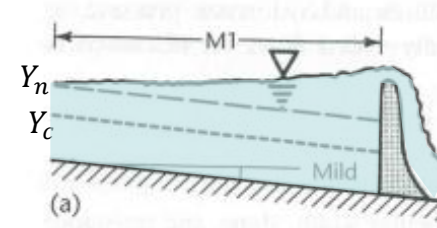
1  
Y above  $Y_n$  &  $Y_c$

2  
Y between  $Y_n$  &  $Y_c$

3  
Y below  $Y_n$  &  $Y_c$

Mild Slopes  
(M)

Steep Slopes  
(S)

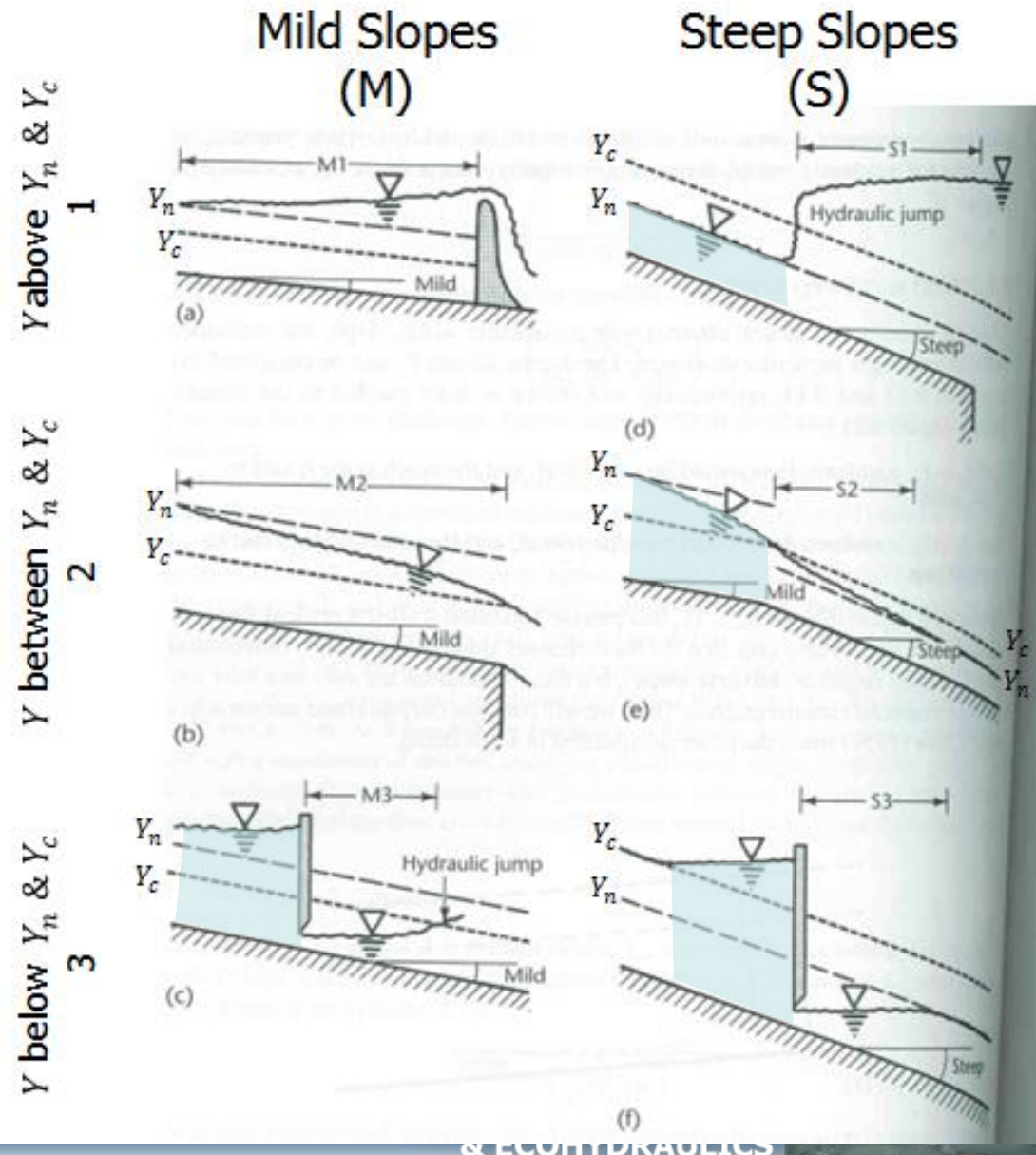


(f)

# NOW THAT YOU KNOW THIS... WHAT IS:

In terms of M/S,  
1,2,3 & Backwater/  
Drawdown

- S3 - Backwater
- M2 - Drawdown
- M1 - Backwater
- S1 - Backwater



# CLASSIFICATION OF FLOW PROFILES

Table 9.1 Classification of flow profiles in natural channels.<sup>a</sup>

Designation	Depth relations	Type	Flow state	Figure
<b>Mild slopes (<math>Y_n &gt; Y_c</math>)</b>				
M1	$Y > Y_n > Y_c$	Backwater; $\frac{dY}{dX} > 0$	Subcritical	9.3a
M2	$Y_n > Y > Y_c$	Drawdown; $\frac{dY}{dX} < 0$	Subcritical	9.3b
M3	$Y_n > Y_c > Y$	Backwater; $\frac{dY}{dX} > 0$	Supercritical	9.3c
<b>Step slopes (<math>Y_c &gt; Y_n</math>)</b>				
S1	$Y > Y_c > Y_n$	Backwater; $\frac{dY}{dX} > 0$	Subcritical	9.3d
S2	$Y_c > Y > Y_n$	Drawdown; $\frac{dY}{dX} < 0$	Supercritical	9.3e
S3	$Y_c > Y_n > Y$	Backwater; $\frac{dY}{dX} > 0$	Supercritical	9.3f

<sup>a</sup>Typical situations inducing the various profile types are shown in figure 9.3.



# TODAY'S PLAN

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## **GRADUALLY VARIED FLOW & WATER SURFACE PROFILES** (Applied to 1D flows)

I. Water Surface Profile Classification (§ 9.2)

**II. Controls (§ 9.3)**

III. Computation of WS Profiles (§ 9.4)

I. Theoretical Basis

II. Standard-Step Approach

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# CONTROLS...

- Normal depth  $Y_n$  for a given discharge is determined by the local channel width  $W$ , slope  $S_0$  and resistance ( $n_M$  or  $\Omega$ )
- $\therefore$  Spatial change in any of these produces a change in depth  $Y$  as the *flow seeks to achieve* the new normal depth  $Y_n$
- A **control** is:

$$Y_n = \left( \frac{n_M \cdot Q}{u_M \cdot W \cdot S_0^{1/2}} \right)^{3/5}$$

Eq. 9.13



normal depth. A **control** is a portion of a channel in which a relatively **marked change occurs in one or more of the factors controlling normal depth** such that it determines the depth associated with a given discharge for some distance along the channel—upstream, downstream, or both. More succinctly, “A control [is] any channel feature, natural or man-made, which **fixes a relationship between depth and discharge in its neighborhood**” (Henderson 1966, p. 174).

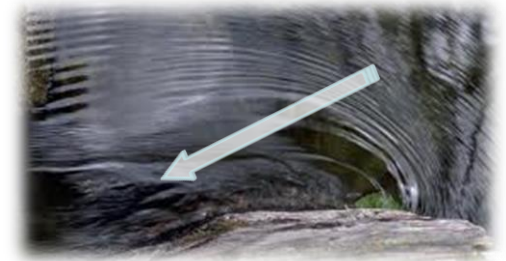


# WAY OF THINKING ABOUT CONTROLS

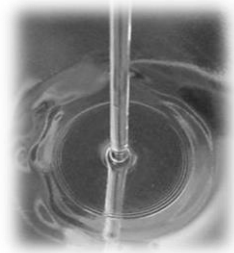
Since controls are basically a  $\frac{dY}{dX}$ :

- A change in depth can be thought of as a positive (DS) or negative (US) gravity wave that travels along the channel at the celerity  $C_{gw}$

$$C_{gw} = \sqrt{g \cdot Y} \quad \text{Eq. 9.15 \& 6.4}$$

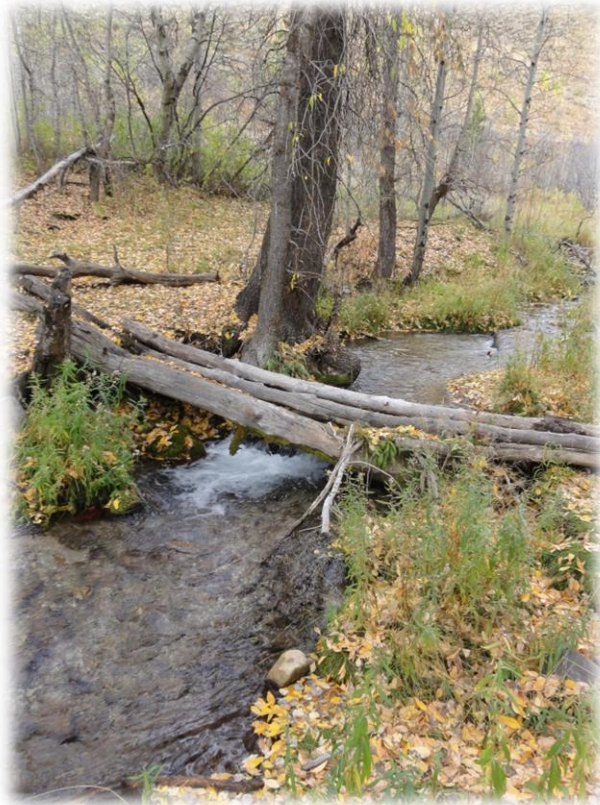


- $C_{gw}$  is wave's velocity with respect to water velocity  $U$
- If flow is subcritical,  $C_{gw} > U$  and depth change propagates both upstream and downstream
- If flow is critical,  $C_{gw} < U$ , "and the 'information' about the new normal depth cannot be transmitted upstream"



# SLOPE CONTROLS (i.e. GRADE BREAKS)

Which are backwaters vs. drawdowns?

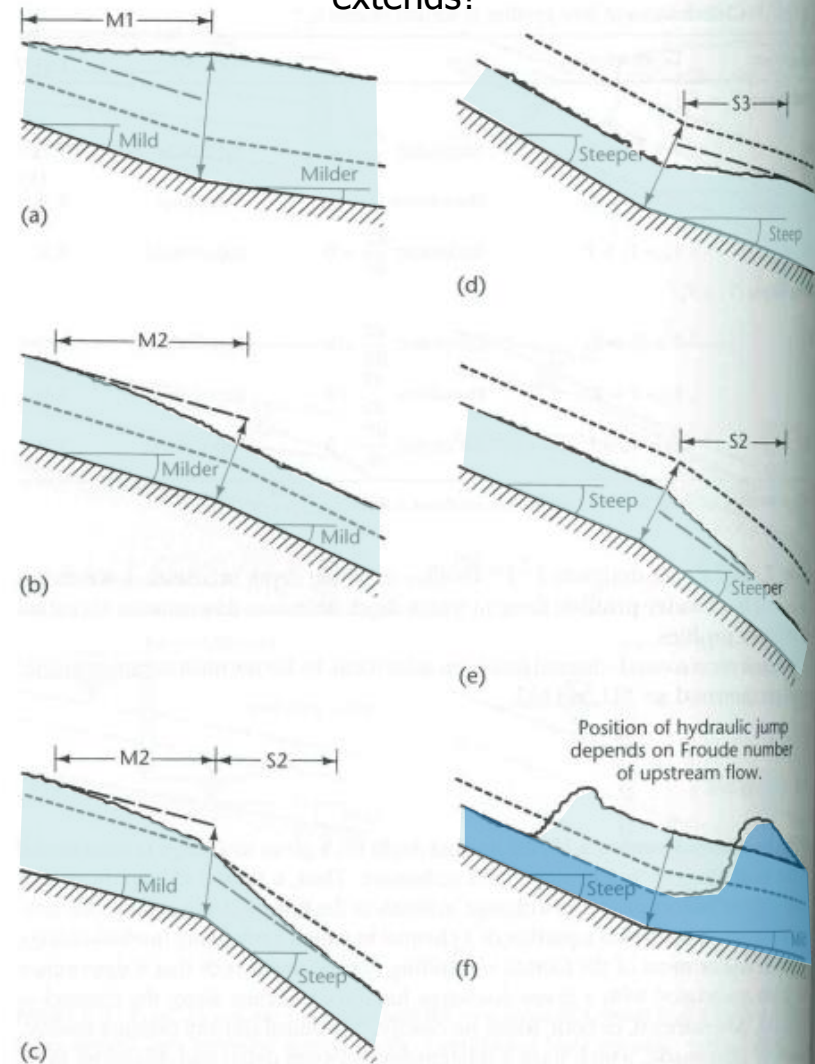


Is  $Y$  always seeking new  $Y_n$  DS of control?

$Fr < 1$  US of Control

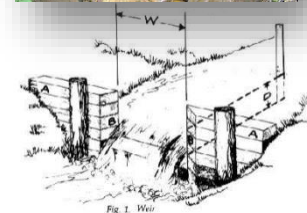
$Fr > 1$  US of Control

Influence extends?



# OTHER TYPES OF CONTROLS

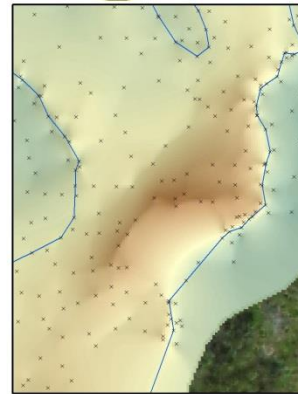
- **Section Controls** - changes in  $W$ ,  $S_0$  and  $n_M$  that produce a control over a relatively short, distinct reach
- **Channel Controls** - changes in  $W$ ,  $S_0$  and  $n_M$  that produce a more diffuse change in  $Y$  over longer distances
- **Partial Controls** – controls that have stage dependence
- **Artificial Controls** – controls designed to provide stable, precise relations between discharge & depth for measuring  $Q$



# WHAT ARE THE CONTROLS?



0 5 10 20 30 Meters



Section controls?  
Channel controls?  
Partial controls?



# TODAY'S PLAN

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# USE SAME STUFF IN TWO DIFFERENT WAYS

$$Q = W_i \cdot Y_i \cdot U_i$$

Eq. 9.1

Estimate

$$Z_i + Y_i + \frac{\alpha_i \cdot U_i^2}{2 \cdot g} = \Delta H + Z_{i-1} + Y_{i-1} + \frac{\alpha_{i-1} \cdot U_{i-1}^2}{2 \cdot g}$$

Eq. 9.2

With:

- Continuity
- Energy
- Uniform-flow resistance



$$Q = u_m \cdot n_{Mi}^{-1} \cdot Y_i \cdot W_i^{5/3} \cdot S_{fi}^{1/2}$$

Eq. 9.12M

Theoretical Derivation  
*(Using continuous mathematics)*

The Standard Step Method  
*(Using discrete mathematics)*



# TODAY'S PLAN

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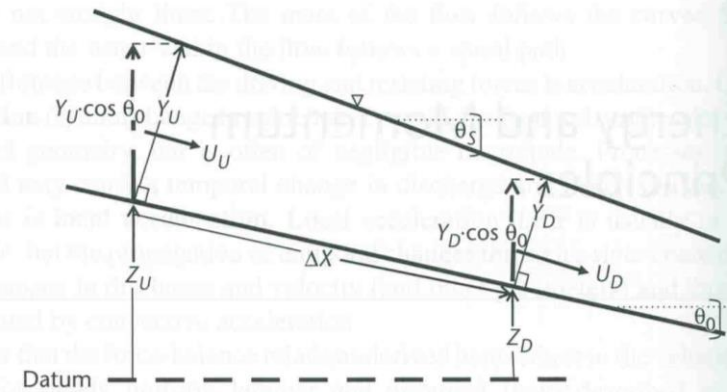
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# START WITH ENERGY...

- Consider a channel carrying steady flow of specified discharge  $Q$ .
- To simplify derivation, assume
  - energy coefficient  $\alpha = 1$
  - Hydrostatic pressure distribution
  - $\cos \theta = 1$
- Recall, definition of specific head  $\mathcal{H}$  (total energy per weight of flowing water) at a given cross section  $\mathcal{H}$  is simply sum of elevation head  $Z$  and specific head  $\mathcal{H}_S$



- X-S Geometry can vary now!
- Change  $U$  &  $D$  subscript to  $i$  and  $i+n$

$$\mathcal{H} = Z + \mathcal{H}_S$$

Eq. 9.16



# LOOK AT $\mathcal{H}$ AS YOU GO DOWNSTREAM

- Since  $\frac{dY}{dX} \neq 0$  take derivative of energy  $\mathcal{H}$  relative to downstream distance  $X$

$$\mathcal{H} = Z + \mathcal{H}_S \quad \longrightarrow \quad \frac{d\mathcal{H}}{dX} = \frac{dZ}{dX} + \frac{d\mathcal{H}_S}{dX} \quad \text{Eq. 9.17}$$

- Recalling definition of channel slope ( $S_0 \equiv -\frac{\Delta Z}{\Delta X}$ ) and friction slope ( $S_f \equiv \left(\frac{Q}{K}\right)^2$ ):

$$\frac{d\mathcal{H}_S}{dX} = S_0 - S_f \quad \text{Eq. 9.18}$$

- And because:

$$\frac{d\mathcal{H}_S}{dX} = \frac{d\mathcal{H}_S}{dY} \cdot \frac{dY}{dX} \quad \text{Eq. 9.19}$$

We can get change in depth as we go downstream:

$$\frac{dY}{dX} = \frac{S_0 - S_f}{\frac{d\mathcal{H}_S}{dY}} \quad \text{Eq. 9.20}$$



# THE DOWNSTREAM RATE OF CHANGE OF DEPTH RELATION

- Towards a more useful form of 9.20:

$$\frac{dY}{dX} = \frac{S_0 - S_f}{\frac{d\mathcal{H}_S}{dX}} \quad \text{Eq. 9.20}$$

## BOX 9.1 Derivation of the Downstream Rate-of-Change-of-Depth Relation (Equation 9.21)

We saw in equation 8.13 that

$$\frac{d\mathcal{H}_S}{dY} = 1 - \frac{Q^2}{g \cdot W^2 \cdot Y^3}, \quad (9B1.1)$$

and substituting equation 9B1.1 into equation 9.20 gives

$$\frac{dY}{dX} = \frac{S_0 - S_f}{1 - \frac{Q^2}{g \cdot W^2 \cdot Y^3}}. \quad (9B1.2)$$

This expression can be simplified by recalling from equation 8.14 that

$$\frac{Q^2}{g \cdot W^2} = Y_c^3, \quad (9B1.3)$$

so equation 9B1.2 can be written as

$$\frac{dY}{dX} = \frac{S_0 - S_f}{1 - \left(\frac{Y_c}{Y}\right)^3}. \quad (9B1.4)$$

We can write the numerator of equation 9B1.4 in a form similar to the denominator by noting that  $S_0 - S_f = S_0 \cdot (1 - S_f/S_0)$ :

$$\frac{dY}{dX} = S_0 \cdot \frac{1 - \left(\frac{S_f}{S_0}\right)}{1 - \left(\frac{Y_c}{Y}\right)^3} \quad (9B1.5)$$



$$\frac{dY}{dX} = S_0 \cdot \left[ \frac{1 - \left(\frac{S_f}{S_0}\right)}{1 - \left(\frac{Y_c}{Y}\right)^3} \right]$$

Eq. 9.21

- Just a function of  $S_0$ ,  $S_f$ ,  $Y_c$ , &  $Y$
- Would be nice to get rid of  $S_f$



# BRING IN FLOW RESISTANCE

- Now we use our uniform flow resistance relationship (just energy up to this point); can be Chezy or Mannings
- In both:
  - Normal depth  $Y_n$  is related to channel slope  $S_0$
  - Actual depth  $Y$  is related to friction slope  $S_f$  (assuming that uniform-flow relations are applicable to gradually varied flow)

- Using Chezy..

$$Y_n = \left( \frac{\Omega \cdot Q}{g^{1/2} \cdot W \cdot S_0^{1/2}} \right)^{2/3}$$



Eq. 9.13C & 9.23

$$S_0 = \frac{\Omega^2 \cdot Q^2}{g \cdot W^2 \cdot Y_n^3}$$

Eq. 9.24

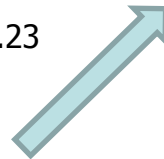
$$\frac{S_f}{S_0} = \left( \frac{Y_n}{Y} \right)^3$$

Eq. 9.25C

Or for Mannings:

$$\frac{S_f}{S_0} = \left( \frac{Y_n}{Y} \right)^{10/3}$$

Eq. 9.25M



# NOW SUBSTITUTING RESISTANCE TO GET $\frac{dY}{dX}$

- Original:

$$\frac{dY}{dX} = S_0 \cdot \left[ \frac{1 - \left( S_f / S_0 \right)}{1 - \left( Y_c / Y \right)^3} \right] \quad \text{Eq. 9.21}$$

- Chezy

$$\frac{dY}{dX} = S_0 \cdot \left[ \frac{1 - \left( Y_n / Y \right)^3}{1 - \left( Y_c / Y \right)^3} \right] \quad \text{Eq. 9.26C}$$

- Mannings

$$\frac{dY}{dX} = S_0 \cdot \left[ \frac{1 - \left( Y_n / Y \right)^{10/3}}{1 - \left( Y_c / Y \right)^{10/3}} \right] \quad \text{Eq. 9.26M}$$



# NOW, WE CAN RELATE $\frac{dY}{dX}$ TO CLASSIFICATION

- Define:

$$N \equiv 1 - \left( Y_n / Y \right)^3 \quad \text{Eq. 9.27C}$$

or:

$$N \equiv 1 - \left( Y_n / Y \right)^{10/3} \quad \text{Eq. 9.27M}$$

For both...

$$D \equiv 1 - \left( Y_c / Y \right)^3 \quad \text{Eq. 9.28}$$

- With N & D
  - If  $Y_n < Y$ , then  $N > 0$
  - If  $Y_c < Y$ , then  $D > 0$
  - If  $Y_c > Y$ , then  $D < 0$

- $\therefore$  the sign of the ratio of  $\frac{N}{D}$  determines the sign of  $\frac{dY}{dX}$  (i.e. whether depth increases or decreases)

Table 9.1 Classification of flow profiles in natural channels.<sup>d</sup>

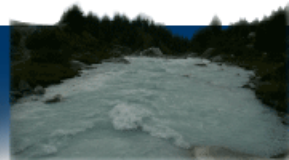
Designation	Depth relations	Type	Flow state	Figure
<b>Mild slopes (<math>Y_n &gt; Y_c</math>)</b>				
M1	$Y > Y_n > Y_c$	Backwater; $\frac{dY}{dX} > 0$	Subcritical	9.3a
M2	$Y_n > Y > Y_c$	Drawdown; $\frac{dY}{dX} < 0$	Subcritical	9.3b
M3	$Y_n > Y_c > Y$	Backwater; $\frac{dY}{dX} > 0$	Supercritical	9.3c
<b>Steep slopes (<math>Y_c &gt; Y_n</math>)</b>				
S1	$Y > Y_c > Y_n$	Backwater; $\frac{dY}{dX} > 0$	Subcritical	9.3d
S2	$Y_c > Y > Y_n$	Drawdown; $\frac{dY}{dX} < 0$	Supercritical	9.3e
S3	$Y_c > Y_n > Y$	Backwater; $\frac{dY}{dX} > 0$	Supercritical	9.3f

<sup>d</sup>Typical situations inducing the various profile types are shown in figure 9.3.



Table 9.2 Relation of water-surface profile classification (table 9.1, figure 9.3) to equations 9.26-9.28, assuming  $S_0 > 0$ .

Depth relations	N	D	$N/D, \frac{dY}{dX}$	Profile type
<b>Mild slopes (<math>Y_n &gt; Y_c</math>)</b>				
$Y > Y_n > Y_c$	$1 > N > 0$	$1 > D > 0$	$> 0$	M1, backwater
$Y_n > Y > Y_c$	$N < 0$	$1 > D > 0$	$< 0$	M2, drawdown
$Y_n > Y_c > Y$	$N < 0$	$D < 0$	$> 0$	M3, backwater
<b>Steep slopes (<math>Y_c &gt; Y_n</math>)</b>				
$Y > Y_c > Y_n$	$1 > N > 0$	$1 > D > 0$	$> 0$	S1, backwater
$Y_c > Y > Y_n$	$1 > N > 0$	$D < 0$	$< 0$	S2, drawdown
$Y_c > Y_n > Y$	$N < 0$	$D < 0$	$> 0$	S3, backwater



# WORTH NOTING...

- Recall specific head diagram

$$H_S = Y + \frac{\alpha_i \cdot Q^2}{2 \cdot g \cdot W^2 \cdot Y^2}$$

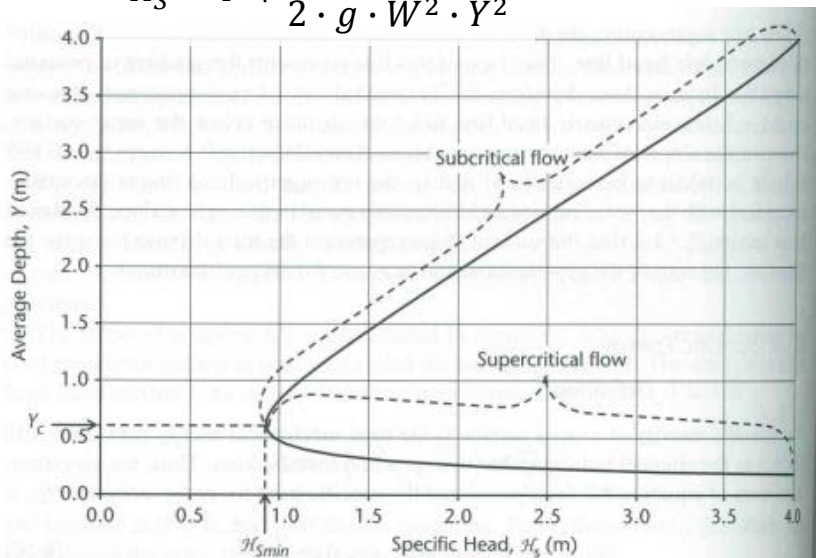


Figure 8.6 A specific-head diagram for a discharge  $Q = 4 \text{ m}^3/\text{s}$  in a channel of width  $W = 3 \text{ m}$ . The curve represents solutions to equation 8.12. The upper limb of the curve are depths for subcritical flows; the lower limb, for supercritical flows. Velocity head is computed assuming  $\alpha = 1.3$ .  $H_{Smin}$  is the minimum possible specific head for this discharge; the corresponding depth is the critical depth,  $Y_c = 0.57 \text{ m}$ .

Table 9.2 Relation of water-surface profile classification (table 9.1, figure 9.3) to equations 9.26–9.28, assuming  $S_0 > 0$ .

Depth relations	$N$	$D$	$N/D, \frac{dY}{dX}$	Profile type
<b>Mild slopes (<math>Y_n &gt; Y_c</math>)</b>				
$Y > Y_n > Y_c$	$1 > N > 0$	$1 > D > 0$	$> 0$	M1, backwater
$Y_c > Y > Y_n$	$N < 0$	$1 > D > 0$	$< 0$	M2, drawdown
$Y_c > Y_n > Y$	$N < 0$	$D < 0$	$> 0$	M3, backwater
<b>Steep slopes (<math>Y_c &gt; Y_n</math>)</b>				
$Y > Y_c > Y_n$	$1 > N > 0$	$1 > D > 0$	$> 0$	S1, backwater
$Y_c > Y > Y_n$	$1 > N > 0$	$D < 0$	$< 0$	S2, drawdown
$Y_c > Y_n > Y$	$N < 0$	$D < 0$	$> 0$	S3, backwater

Two other implications of equation 9.26 are of interest. When  $Y = Y_n$ ,  $dY/dX = 0$ , consistent with the fact that depth does not change in a reach with uniform flow. However, when  $Y = Y_c$ , the change in depth is not defined. This reflects the fact that water surfaces are unstable when flows are near critical, as discussed in section 6.2.2.2. This instability is also suggested in the specific-head diagram (see figure 8.6), which has a very steep slope in the vicinity of the critical depth. This means that when the flow is near the critical regime, a small change in its energy leads to a relatively large depth change. In natural channels, there are ubiquitous small variations in slope, width, and resistance that affect the energy of the flow, so when the flow is near critical, the surface is wavy and irregular (figure 9.6). Under these conditions, the flow is rapidly varied rather than gradually varied, and the assumptions of uniform flow are no longer valid.

Equation 9.26 can be rearranged to

$$dY = S_0 \cdot \left[ \frac{1 - (Y_n/Y)^{10/3}}{1 - (Y_c/Y)^{10/3}} \right] \cdot dX \quad \text{Eq. 9.29}$$



# THEN THE MEANINGFUL BIT...

- Since  $Y_n$  &  $Y_c$  are  $f(dX)$ , we can integrate equation 9.29 between a location  $X_i$  where depth is  $Y_i$  and a location  $X_{i+1}$  where depth is  $Y_{i+1}$ :

$$dY = S_0 \cdot \left[ \frac{1 - (Y_n/Y)^{10/3}}{1 - (Y_c/Y)^{10/3}} \right] \cdot dX \quad \text{Eq. 9.29M}$$

$$\int_{Y_i}^{Y_{i+1}} dY = \int_{Y_i}^{Y_{i+1}} S_0 \cdot \left[ \frac{1 - (Y_n(X)/Y(X))^{10/3}}{1 - (Y_c(X)/Y(X))^{10/3}} \right] \cdot dX$$

$$Y_{i+1} = Y_i + \int_{Y_i}^{Y_{i+1}} S_0 \cdot \left[ \frac{1 - (Y_n(X)/Y(X))^{10/3}}{1 - (Y_c(X)/Y(X))^{10/3}} \right] \cdot dX \quad \text{Eq. 9.30M}$$



# HOW CAN I USE THAT?

$$Y_{i+1} = Y_i + \int_{Y_i}^{Y_{i+1}} S_0 \cdot \left[ \frac{1 - (Y_n(X)/Y(X))^{10/3}}{1 - (Y_c(X)/Y(X))^{10/3}} \right] \cdot dX \quad \text{Eq. 9.30M}$$

- Well...

Again, the exponent in the numerator = 10/3 if the Manning relation is used.) If, for a given discharge, we know 1) the depth at a starting location ( $i = 0$ ), 2) the bottom elevation and channel geometry at successive locations along the channel, and 3) information required for estimating resistance ( $\Omega$  or  $n_M$ ) at successive locations,

equation 9.30 can be integrated numerically to provide successive depths and water-surface elevations. As noted above, if the flow is subcritical, the integration proceeds in the upstream direction, and if supercritical, it proceeds in the downstream direction.

Chow (1959) described various mathematical approaches to integrating equation 9.30. However, in practice, the integration is usually carried out by a finite-difference approach, called the standard step method, described in the following subsection. This method is incorporated, with many elaborations, in computer programs for calculating water-surface profiles, such as the widely used U.S. Army Corps of Engineers' Hydrologic Engineering Center River Analysis System (HEC-RAS; Brunner 2001a).

## Numerically....



# TODAY'S PLAN

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## **GRADUALLY VARIED FLOW & WATER SURFACE PROFILES** (Applied to 1D flows)

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- III. Computation of WS Profiles (§ 9.4)
  - I. Theoretical Basis
  - II. Standard-Step Approach**
- IV. Lab: Using WS Spreadsheet



# A VERY MANUAL FORM OF THE STANDARD STEP

## 9.4.2 The Standard Step Method

### 9.4.2.1 Basic Approach

From equation 8.7, the total mechanical energy per unit weight (head) at cross section  $\mathcal{H}_i$ , can be written as the sum of the potential-energy head,  $\mathcal{H}_{PEi}$ , and the kinetic energy (velocity) head,  $\mathcal{H}_{KEi}$ :

$$\mathcal{H}_i = \mathcal{H}_{PEi} + \mathcal{H}_{KEi}. \quad (9.31)$$

- Again:
  - Continuity
  - Energy
  - Uniform-flow resistance

The potential-energy head represents the elevation of the water surface above a datum (see figure 8.5) (assuming, as we will throughout this section, that  $\cos \theta_0 = 1$ ). Thus, from equation 8.8b, we can write the energy equation between an upstream section (designated by subscript  $i$ ) and a downstream section (designated by subscript  $i - 1$ ) as

$$\mathcal{H}_{PEi} + \mathcal{H}_{KEi} = \mathcal{H}_{PEi-1} + \mathcal{H}_{KEi-1} + \Delta\mathcal{H}, \quad (9.32)$$

where  $\Delta\mathcal{H}$  is the total head loss between the two sections. For subcritical flow we compute in the upstream direction, so the working form of equation 9.32 is

$$\mathcal{H}_{PEi} = \mathcal{H}_{PEi-1} + \mathcal{H}_{KEi-1} + \Delta\mathcal{H} - \mathcal{H}_{KEi}. \quad (9.33a)$$

For supercritical flow, we solve for the downstream water-surface elevation:

$$\mathcal{H}_{PEi-1} = \mathcal{H}_{PEi} + \mathcal{H}_{KEi} - \Delta\mathcal{H} - \mathcal{H}_{KEi-1}. \quad (9.33b)$$

Since subcritical flow is by far the more common, subsequent developments here will use only equation 9.33a.

Following the discussion in section 8.3, the total head loss between sections is usually divided into two parts:

$$\Delta\mathcal{H} = \Delta\mathcal{M} + \Delta\mathcal{H}_{\text{eddy}}, \quad (9.34)$$

where  $\Delta\mathcal{M}$  represents the energy loss due to friction with the flow boundary (**friction loss**), and  $\Delta\mathcal{H}_{\text{eddy}}$  represents the energy losses due to flow expansion or contraction (**eddy loss** or **shock loss**). The friction loss is computed from the average friction slope  $\bar{S}_f$ , which is computed from the selected uniform-flow equation at the upstream and downstream sections:

$$\frac{\Delta\mathcal{M}}{\Delta X} \equiv \bar{S}_f, \quad (9.35)$$

$$\Delta\mathcal{M} = \bar{S}_f \cdot \Delta X.$$

The eddy loss is usually estimated via equation 9.5b as

$$\Delta\mathcal{H}_{\text{eddy}} = k_{\text{eddy}} \cdot \left( \frac{U_i^2}{2 \cdot g} - \frac{U_{i-1}^2}{2 \cdot g} \right), \quad (9.36)$$

where  $k_{\text{eddy}}$  is estimated as described in table 9.3.

Combining equations 9.33a and 9.34, the basic working equation for computing water-surface profiles in subcritical flows is

$$\mathcal{H}_{PEi} = \mathcal{H}_{PEi-1} + \mathcal{H}_{KEi-1} - \mathcal{H}_{KEi} + \Delta\mathcal{M} + \Delta\mathcal{H}_{\text{eddy}}, \quad (9.37)$$

from which the upstream depth,  $Y_i$ , is calculated as the difference between the potential head and the bed elevation,  $Z_i$ :

$$Y_i = \mathcal{H}_{PEi} - Z_i \quad (9.38)$$

Table 9.3 Values of the eddy-loss coefficient  $k_{\text{eddy}}$  for subcritical flows (after Brunner 2001b).

Nature of width transition	Contraction ( $W_U > W_D$ )	Expansion ( $W_U < W_D$ )
None to very gradual	0.0	0.0
Gradual	0.1	0.3
Typical bridge sections	0.3	0.5
Abrupt	0.6	0.8

# LET'S DISCUSS & PLAY WITH THE REST

Table 9.4 Example of water-surface profile computations.

1	2	3	4	5	6	7	8	9	10	11
Sect., <i>i</i>	Distance, <i>X</i> (m)	Bed elev., <i>Z<sub>0</sub></i> (m)	Width, <i>W</i> (m)	Manning <i>n<sub>M</sub></i>	Est. depth, <i>Ŷ</i> (m)	Area, <i>A</i> (m <sup>2</sup> )	Velocity, <i>U</i> (m/s)	Froude no., <i>Fr</i>	Vel. head, <i>ŷ<sub>KE</sub></i> (m)	Fric. slope, <i>S<sub>f</sub></i>
0	0	843.14	147.0	0.043	10.21	1500.85	2.67	0.27	4.71E-01	5.93E-04
1	100	843.25	137.2	0.036	10.16	1393.95	2.87	0.29	5.46E-01	4.85E-04
2	200	843.00	152.4	0.039	10.73	1624.58	2.46	0.24	4.02E-01	3.93E-04
3	400	844.05	162.8	0.037	9.67	1574.28	2.54	0.26	4.28E-01	4.29E-04
4	600	844.57	162.0	0.045	9.26	1500.12	2.67	0.28	4.71E-01	7.43E-04
5	1000	845.81	167.3	0.040	8.48	1418.70	2.82	0.31	5.27E-01	7.36E-04
6	1500	846.74	128.2	0.051	8.36	1071.75	3.73	0.41	9.23E-01	2.14E-03
7	2000	847.23	150.3	0.038	9.05	1360.22	2.94	0.31	5.73E-01	6.27E-04
8	2500	849.00	161.0	0.043	7.45	1199.45	3.34	0.39	7.37E-01	1.41E-03

See text for discussion. The computed profile is plotted in figure 9.8. Pot., potential.

### 9.4.2.2 Detailed Steps and Example Calculation

Here we describe the details and show the results of an example computation using the standard step method. The procedure used here, based on computations carried out via the spreadsheet program WSPROFILE.XLS (available at the book's website, <http://www.oup.com/us/fluvialhydraulics>; see appendix D), is a much-simplified version of the approach incorporated in such programs as the U.S. Army Corps of Engineers' HEC-RAS (Brunner 2001a, 2001b) or the U.S. Geological Survey's WSPRO (for Water-Surface Profile) program (Shearman 1990). HEC-RAS is a very elaborate but user-friendly program that is widely used by practitioners for calculating water-surface profiles.

The computations can be followed in table 9.4. In this (fictitious) example, we calculate the water-surface profile for a river upstream of its entrance into a reservoir when the discharge is 4,000 m<sup>3</sup>/s. The channel characteristics determined by survey and observation are entered in columns 2–5. The depth at the downstream end (*Y<sub>0</sub>*) is fixed by the known reservoir elevation, which has been entered in the first row of column 6. At all sections, we assume a rectangular channel with  $\alpha = 1.3$  and  $k_{\text{eddy}} = 0$  for expanding sections and 0.1 for contracting sections. We specify a tolerance of  $\Delta Y = 0.02$  m as the maximum acceptable difference between the initial trial depth and the computed depth.

Once *Y<sub>0</sub>* is entered, the other quantities for that section (except slope) are calculated in other columns. Beginning at the next upstream section (*i* = 1), computation proceeds via the following steps. Quantities that have been predetermined by survey, observation, or estimation are shown in boldface:

Column 6. Enter a trial depth  $\hat{Y}_i$ .

Column 7. The cross-sectional area of the flow is computed as  $A_i = \hat{Y}_i \cdot W_i$ .

12	13	14	15	16	17	18	19	20	21
Av. slope, <i>S<sub>f</sub></i>	Fric. loss, $\Delta \mathcal{M}$ (m)	Eddy loss, $\Delta \mathcal{H}_{\text{eddy}}$ (m)	Total head, <i>H</i> (m)	Pot. head, <i>H<sub>PE</sub></i> (m)	Pot. head OK?	Calc. depth, <i>Y</i> (m)	Depth OK?	Normal depth, <i>Y<sub>n</sub></i> (m)	Critical, <i>Y<sub>c</sub></i> (m)
			853.82			10.21		8.48	4.23
12E-04	5.39E-02	7.50E-03	853.88	853.40	yes	10.15	yes	7.95	4.43
13E-04	4.39E-02	4.32E-02	853.97	853.65	yes	10.65	yes	6.12	4.13
14E-04	8.22E-02	7.82E-03	854.06	853.72	yes	9.67	yes	4.56	3.95
15E-04	1.17E-01	4.33E-03	854.18	853.83	yes	9.26	yes	6.35	3.96
16E-04	2.95E-01	1.67E-02	854.49	854.28	yes	8.47	yes	5.57	3.88
17E-03	7.18E-01	3.96E-02	855.25	855.08	yes	8.34	yes	8.71	4.63
18E-03	6.99E-01	1.05E-01	856.05	856.29	yes	9.06	yes	8.05	4.16
19E-03	5.19E-01	4.92E-02	856.62	856.45	yes	7.45	yes	5.66	3.98

Column 8. The cross-sectional average velocity is computed as  $U_i = Q/A_i$ .

Column 9. The Froude number is computed as  $Fr_i = U_i/(g \cdot \hat{Y}_i)^{1/2}$ . This provides a check that the flow is subcritical ( $Fr_i < 1$ ) so that computations can proceed in the upstream direction.

Column 10. The velocity head is computed as  $\mathcal{H}_{KEi} = \alpha_i \cdot U_i^2/(2 \cdot g)$ .

Column 11. The friction slope  $S_{fi}$  is computed. Here we use the Manning equation, so  $S_{fi} = n_{Mi}^2 \cdot Q^2 / (u_m^2 \cdot W_i^2 \cdot \hat{Y}_i^{10/3})$ . We assume a single value of  $n_{Mi}$  at each cross section, but in many sections the resistance varies significantly as a function of width, especially if floodplains are included, and this variation must be accounted for. Box 9.2 and figure 9.7 describe the general approach for doing this. (The example assumes no overbank flow or cross-sectional variation of resistance.)

Column 12. The average friction slope for adjacent sections,  $\bar{S}_{fi}$ , is determined as the arithmetic mean of the slope at the current section and the adjacent downstream section:  $\bar{S}_{fi} = (S_{fi-1} + S_{fi})/2$ .

Column 13. The friction loss between sections *i* and *i* - 1,  $\Delta \mathcal{M}_{i,i-1}$ , is calculated as the product of  $\bar{S}_{fi}$  and the distance between the two sections:  $\Delta \mathcal{M}_{i,i-1} = \bar{S}_{fi} \cdot (X_i - X_{i-1})$ .

Column 14. The eddy loss is computed as  $\Delta \mathcal{H}_{\text{eddy},i,i-1} = k_{\text{eddy}} | [U_i^2/(2 \cdot g) - U_{i-1}^2/(2 \cdot g)] |$ .

Column 15. The total head  $\mathcal{H}_i$  is computed as  $\mathcal{H}_i = \mathcal{H}_{i-1} + \Delta \mathcal{H}_{i,i-1} = \mathcal{H}_{i-1} + \Delta \mathcal{M}_{i,i-1} + \Delta \mathcal{H}_{\text{eddy},i,i-1}$ . To satisfy the second law of thermodynamics, it must be true that  $\mathcal{H}_i > \mathcal{H}_{i-1}$  where section *i* is upstream of section *i* - 1.

Column 16. The potential head  $\mathcal{H}_{PEi}$  is computed as  $\mathcal{H}_{PEi} = \mathcal{H}_{i-1} - \mathcal{H}_{KEi} + \Delta \mathcal{M}_{i,i-1} + \Delta \mathcal{H}_{\text{eddy},i,i-1}$ .

Column 17. Here we check that  $\mathcal{H}_{PEi} > \mathcal{H}_{PEi-1}$ , which must be true in order for flow to occur. "No" appears in this column if this condition is not satisfied.



# TODAY'S PLAN

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## **IV. Lab: Using WS Spreadsheet**

