The Hydraulics of Open Channel Flow: An Introduction

Basic principles, sediment motion, hydraulic modelling, design of hydraulic structures

Second Edition

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Summary

In this chapter, the fundamental equations are developed and applied to open channel flows. It is shown that the momentum equation leads to the Bernoulli equation.

2.1 INTRODUCTION

In open channel flow the free surface is always at a constant absolute pressure (usually atmospheric) and the driving force of the fluid motion is gravity. In most practical situations, open channels contain waters. The general principles of open channel flow calculations developed in this chapter are, however, applicable to other liquids. Specific results (e.g. flow resistance) are based primarily upon experimental data obtained mostly with water.

2.2 THE FUNDAMENTAL EQUATIONS

2.2.1 Introduction

The law of conservation of mass states that the mass within a closed system remains constant with time (disregarding relativity effects):

$$\frac{DM}{Dt} = 0$$  \hspace{1cm} (2.1)

where $M$ is the total mass and $D/\text{Dt}$ is the absolute differential (see Appendix A1.3, section on Differential and differentiation). Equation (2.1) leads to the continuity equation.

The expression of Newton's second law of motion for a system is:

$$\sum \vec{F} = \frac{D}{Dt} \left( M \times \vec{V} \right)$$  \hspace{1cm} (2.2)

where $\sum \vec{F}$ refers to the resultant of all external forces acting on the system including body forces such as gravity, and $\vec{V}$ is the velocity of the centre of mass of the system. The application of equation (2.2) is called the motion equation.

Equations (2.1) and (2.2) must be applied to a control volume. A control volume is a specific region of space selected for analysis. It refers to a region in space where fluid enters and leaves (e.g. Fig. 2.1). The boundary of a control volume is its control surface. The concept of control volume used in conjunction with the differential form of the continuity, momentum and energy equations, is an open system.
All flow situations including open channel flows are subject to the following relationships (e.g. Streeter and Wylie, 1981: pp. 88–89):

1. the first and second laws of thermodynamics,
2. the law of conservation of mass (i.e. the continuity relationship),
3. Newton’s law of motion,
4. the boundary conditions.

Other relations (such as a state equation, Newton’s law of viscosity) may apply.

**Note**
A control volume may be either infinitesimally small or finite. It may move or remain fixed in space. It is an *imaginary* volume and *does not interfere* with the flow.

### 2.2.2 The continuity equation

The principle of conservation of mass states that the mass within a closed system remains constant with time:

\[
\frac{DM}{Dt} = \frac{D}{Dt} \iiint_{V} \rho \mathrm{d}V = 0
\]

(2.3)

where \( \rho \) is the fluid density and \((x, y, z)\) are three components of a Cartesian system of co-ordinates.

For an infinitesimal small control volume the continuity equation is:

\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{V}) = 0
\]

(2.4a)

and in Cartesian co-ordinates:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V_x)}{\partial x} + \frac{\partial (\rho V_y)}{\partial y} + \frac{\partial (\rho V_z)}{\partial z} = 0
\]

(2.4b)

where \( V_x, V_y \) and \( V_z \) are the velocity components in the \( x-, y- \) and \( z- \) directions, respectively.
For an incompressible flow (i.e. \( \rho = \text{constant} \)) the continuity equation becomes:

\[
\text{div} \, \vec{V} = 0
\]  
(2.5a)

and in Cartesian co-ordinates:

\[
\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0
\]  
(2.5b)

For an incompressible fluid (e.g. open channel flow), the inflow (i.e. amount of fluid entering into a control volume) equals the outflow.

**Application**

Considering an open channel flow with no flow across the side and bottom walls, equation (2.5) can be integrated between two cross-sections of areas \( A_1 \) and \( A_2 \). It yields:

\[
Q = \int_{A_1} V \, dA = \int_{A_2} V \, dA
\]

where \( Q \) is the total discharge (i.e. volume discharge). Integration of the equation leads to:

\[
Q = V_1 A_1 = V_2 A_2
\]

where \( V_1 \) and \( V_2 \) are the mean velocities across the cross-sections \( A_1 \) and \( A_2 \) respectively.

### 2.2.3 The momentum equation

**The Navier–Stokes equation**

Newton's law of motion is used as a basis for developing the momentum equation for a control volume:

\[
\sum F = \frac{\text{D}}{\text{Dt}} \left( M \times \vec{V} \right) = \frac{\partial}{\partial t} \left( \int_{CV} \rho \vec{V} \, d\text{Volume} \right) + \int_{CS} \rho \dot{\vec{V}} \, d\text{Area}
\]  
(2.6)

where \( CV \) and \( CS \) refer to the control volume and control surface, respectively. Basically, it states that the change of momentum equals the sum of all forces applied to the control volume.

The forces acting on the control volume are (1) the surface forces (i.e. pressure and shear forces) acting on the control surface and (2) the volume force (i.e. gravity) applied at the centre of mass of the control volume. For an infinitesimal small volume the momentum equation applied to the \( i \)-component of the vector equation is:

\[
\frac{\text{D}}{\text{Dt}} \left( \rho V_i \right) = \left( \frac{\partial \left( \rho V_i \right)}{\partial t} + \sum_j \left( \rho V_j \frac{\partial V_i}{\partial x_j} \right) \right) = \rho F_{\text{vol},i} + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}
\]  
(2.7)

where \( \text{D}/\text{Dt} \) is the absolute differential (or absolute derivative, see Appendix A1.3), \( V_i \) is the velocity component in the \( i \)-direction, \( F_{\text{vol},i} \) is the resultant of the volume forces (per unit volume), and \( \sigma_{ij} \) is the stress tensor (see notes below). The subscripts \( i \) and \( j \) refer to the Cartesian co-ordinate components (e.g. \( x, y \)).
Fundamental equations

If the volume forces $\mathbf{F}_{\text{vol}}$ derives from a potential $U$, they can be rewritten as: $\mathbf{F}_{\text{vol}} = -\nabla U$ (e.g., gravity force $\mathbf{F}_{\text{vol}} = -\nabla (gz)$). Further, for a Newtonian fluid the stress forces are (1) the pressure forces and (2) the resultant of the viscous forces on the control volume. Hence for a Newtonian fluid and for volume force deriving from a potential, the momentum equation becomes:

$$\frac{D(\rho \mathbf{v})}{Dt} = \rho \mathbf{F}_{\text{vol}} - \nabla P + \mathbf{F}_{\text{visc}}$$  \hspace{1cm} (2.8)

where $P$ is the pressure and $\mathbf{F}_{\text{visc}}$ is the resultant of the viscous forces (per unit volume) on the control volume.

In Cartesian co-ordinates $(x, y, z)$:

$$\left( \frac{\partial (\rho V_x)}{\partial t} + \sum_j V_j \frac{\partial (\rho V_x)}{\partial x_j} \right) = \rho F_{\text{vol},x} - \frac{\partial P}{\partial x} + F_{\text{visc},x}$$  \hspace{1cm} (2.9a)

$$\left( \frac{\partial (\rho V_y)}{\partial t} + \sum_j V_j \frac{\partial (\rho V_y)}{\partial x_j} \right) = \rho F_{\text{vol},y} - \frac{\partial P}{\partial y} + F_{\text{visc},y}$$  \hspace{1cm} (2.9b)

$$\left( \frac{\partial (\rho V_z)}{\partial t} + \sum_j V_j \frac{\partial (\rho V_z)}{\partial x_j} \right) = \rho F_{\text{vol},z} - \frac{\partial P}{\partial z} + F_{\text{visc},z}$$  \hspace{1cm} (2.9c)

where the subscript $j$ refers to the Cartesian co-ordinate components (i.e., $j = x, y, z$). In equation (2.9), the term on the left side is the sum of the momentum accumulation $\partial (\rho V) / \partial t$ plus the momentum flux $\nabla (\rho V) \cdot \mathbf{v}$. The left term is the sum of the forces acting on the control volume: body force (or volume force) acting on the mass as a whole and surface forces acting at the control surface.

For an incompressible flow (i.e., $\rho = \text{constant}$), for a Newtonian fluid and assuming that the viscosity is constant over the control volume, the motion equation becomes:

$$\rho \left( \frac{\partial V_x}{\partial t} + \sum_j V_j \frac{\partial V_x}{\partial x_j} \right) = \rho F_{\text{vol},x} - \frac{\partial P}{\partial x} + F_{\text{visc},x}$$  \hspace{1cm} (2.10a)

$$\rho \left( \frac{\partial V_y}{\partial t} + \sum_j V_j \frac{\partial V_y}{\partial x_j} \right) = \rho F_{\text{vol},y} - \frac{\partial P}{\partial y} + F_{\text{visc},y}$$  \hspace{1cm} (2.10b)

$$\rho \left( \frac{\partial V_z}{\partial t} + \sum_j V_j \frac{\partial V_z}{\partial x_j} \right) = \rho F_{\text{vol},z} - \frac{\partial P}{\partial z} + F_{\text{visc},z}$$  \hspace{1cm} (2.10c)

where $\rho$, the fluid density, is assumed constant in time and space. Equation (2.10) is often called the Navier–Stokes equation.

Considering a two-dimensional flow in the $(x, y)$ plane and for gravity forces, the Navier–Stokes equation becomes:

$$\rho \left( \frac{\partial V_x}{\partial t} + V_y \frac{\partial V_x}{\partial x} + V_x \frac{\partial V_x}{\partial y} \right) = -\rho g \frac{\partial z}{\partial x} - \frac{\partial P}{\partial x} + F_{\text{visc},x}$$  \hspace{1cm} (2.11a)
where $z$ is aligned along the vertical direction and positive upward. Note that the $x$- and $y$-directions are perpendicular to each other and they are independent of (and not necessarily orthogonal to) the vertical direction.

### Notes
1. For gravity force the volume force potential $U$ is:

$$ U = \vec{g} \times \vec{x} $$

where $\vec{g}$ is the gravity acceleration vector and $\vec{x} = (x, y, z)$, $z$ being the vertical direction positive upward. It yields that the gravity force vector equals:

$$ \vec{F}_{vol} = -\text{grad}(gx) $$

2. A Newtonian fluid is characterized by a linear relation between the magnitude of shear stress $\tau$ and the rate of deformation $\partial V / \partial y$ (equation (1.1)), and the stress tensor is:

$$ \sigma_{ij} = -P\delta_{ij} + \tau_{ij} $$

$$ \tau_{ij} = -\frac{2\mu}{3} \varepsilon \delta_{ij} + 2\mu \varepsilon_{ij} $$

where $P$ is the static pressure, $\tau_{ij}$ is the shear stress component of the $i$-momentum transported in the $j$-direction, $\delta_{ij}$ is the identity matrix element: $\delta_{ii} = 1$ and $\delta_{ij} = 0$ (for $i$ different from $j$),

$$ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) $$

and

$$ \varepsilon = \text{div} \vec{V} = \sum_i \frac{\partial V_i}{\partial x_i} $$

3. The vector of the viscous forces is:

$$ \vec{F}_{\text{visc}} = \text{div} \vec{\tau} = \sum_j \frac{\partial \tau_{ij}}{\partial x_j} $$

For an incompressible flow the continuity equation gives: $\varepsilon = \text{div} \vec{V} = 0$. And the viscous force per unit volume becomes:

$$ \vec{F}_{\text{visc}} = \sum_j \mu \frac{\partial^2 V_j}{\partial x_j \partial x_j} $$

where $\mu$ is the dynamic viscosity of the fluid. Substituting this into equation (2.11), yields:

$$ \rho \left( \frac{\partial V_x}{\partial t} + V_z \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} \right) = -\rho g \frac{\partial z}{\partial x} - \frac{\partial P}{\partial x} + \sum_j \mu \frac{\partial^2 V_x}{\partial x_j \partial x_j} $$

(2.12a)

$$ \rho \left( \frac{\partial V_y}{\partial t} + V_z \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} \right) = -\rho g \frac{\partial z}{\partial y} - \frac{\partial P}{\partial y} + \sum_j \mu \frac{\partial^2 V_y}{\partial x_j \partial x_j} $$

(2.12b)

Equation (2.12) is the original Navier–Stokes equation.
4. Equation (2.12) was first derived by Navier in 1822 and Poisson in 1829 by an entirely different method. Equations (2.10)–(2.12) were derived later in a manner similar as above by Barré de Saint-Venant in 1843 and Stokes in 1845.

5. Louis Navier (1785–1835) was a French engineer who not only primarily designed bridge but also extended Euler’s equations of motion. Siméon Denis Poisson (1781–1840) was a French mathematician and scientist. He developed the theory of elasticity, a theory of electricity and a theory of magnetism. Adhémar Jean Claude Barré de Saint-Venant (1797–1886), French engineer, developed the equations of motion of a fluid particle in terms of the shear and normal forces exerted on it. George Gabriel Stokes (1819–1903), British mathematician and physicist, is known for his research in hydrodynamics and a study of elasticity.

**Application**

Considering an open channel flow in a rectangular channel (Fig. 2.2), we assume a one-dimensional flow, with uniform velocity distribution, a constant channel slope \( \theta \) and a constant channel width \( B \).

The Navier–Stokes equation in the \( s \)-direction is:

\[
\rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \right) = -\rho g \frac{\partial z}{\partial s} - \frac{\partial P}{\partial s} + F_{\text{visc}}
\]

where \( v \) is the velocity along a streamline. Integrating the Navier–Stokes equation over the control volume (Fig. 2.2), the forces acting on the control volume shown in Fig. 2.2 in the \( s \)-direction are:

\[
\int_{CV} -\rho g \frac{dz}{ds} = +\rho g A \Delta s \sin \theta \quad \text{Volume force (i.e. weight)}
\]

\[
\int_{CV} -\frac{dP}{ds} = -\rho g d \Delta B \cos \theta \quad \text{Pressure force (assuming hydrostatic pressure distribution)}
\]

\[
\int_{CV} F_{\text{visc}} = -\tau_{\text{av}} P_{\text{w}} \Delta s \quad \text{Friction force (i.e. boundary shear)}
\]

where \( A \) is the cross-sectional area (i.e. \( A = Bd \) for a rectangular channel), \( d \) is the flow depth, \( \Delta s \) is the length of the control volume, \( \tau_{\text{av}} \) is the average bottom shear stress and \( P_{\text{w}} \) is the wetted perimeter.

![Fig. 2.2 Control volume for an open channel flow.](image)
Assuming a steady flow, the change in momentum equals:

$$\int_{S} \rho \frac{dV}{ds} = \rho A \Delta s \frac{\Delta V}{\Delta s}$$

where $V$ is the mean flow velocity. The term $(\rho V)$ is the momentum per unit volume.

The integration of the Navier–Stokes equation for a one-dimensional steady open channel flow yields:

$$\rho A V \Delta V = +p g A \Delta s \sin \theta - p g d \Delta B \cos \theta - \tau_w A \Delta s$$

The term on the left is the gradient of momentum flux: $\Delta \{ (1/2) (\rho V) (V A) \}$ (i.e. the rate of change of momentum). On the right side of the equation, $(+p g A \sin \theta \Delta s)$ is the gravity force (potential energy), $(-p g d \Delta B \cos \theta)$ is the pressure term (work of the flow) and $(-\tau_w A \Delta s)$ is the friction force (losses).

**Application of the momentum equation**

Considering an arbitrary control volume, it is advantageous to select a volume with a control surface perpendicular to the flow direction denoted $s$. For a steady and incompressible flow the forces acting on the control volume in the $s$-direction are equal to the rate of change in the flow momentum (i.e. no momentum accumulation). The momentum equation gives:

$$\sum F_s = \rho \sum A \frac{V_{s2} V_s}{V_{s1}} - \rho_1 A_1 \frac{V_1}{V_{s1}}$$  \hspace{1cm} (2.13a)

where $\sum F_s$ is the resultant of all the forces in the $s$-direction, the subscripts 1 and 2 refer to the upstream and downstream cross-sections, respectively, and $V_{s1}, V_{s2}$ are the velocity component in the $s$-direction. Combining with the continuity equation for a steady and incompressible flow, it yields:

$$\sum F_s = \rho Q (V_{s2} - V_{s1})$$  \hspace{1cm} (2.13b)

In simple terms, the momentum equation states that the change in momentum flux is equal to the sum of all forces (volume and surface forces) acting on the control volume.

**Note**

For a steady incompressible flow, the momentum flux equals $(\rho Q V)$.

**Application: hydraulic jump**

In open channels, the transition from a rapid flow to a slow flow is called a hydraulic jump (Fig. 2.3, Plate 25). The transition occurs suddenly and is characterized by a sudden rise of liquid surface.

The forces acting on the control volume are the hydrostatic pressure forces at each end of the control volume, the gravity force (i.e. the weight of water) and the bottom friction. Considering a horizontal rectangular open channel of constant channel width $B$, and neglecting the shear stress at the channel bottom, the resultant of the forces acting in the $s$-direction are the result of hydrostatic pressure at the ends of the control volume. The continuity equation and momentum...
Fundamental equations

\[ V_1 d_1 B = V_2 d_2 B \]  
\[ \rho Q (\beta_2 V_2 - \beta_1 V_1) = \left( \frac{1}{2} \rho g d_1^2 B_1 \right) - \left( \frac{1}{2} \rho g d_2^2 B_2 \right) \]

where \( B \) is the channel width (assumed constant) and \( Q \) is the total discharge (i.e. \( Q = VdB \)).

**Notes**

If the velocity distribution is not uniform over the cross-section, the average velocity can be used by introducing the momentum correction factor \( \beta \) defined as:

\[ \beta = \frac{\int_A \rho V^2 \, dA}{\rho V^2 A} \]

where \( A \) is the cross-sectional area (normal to the flow direction) and \( V \) is the mean flow velocity (\( V = Q/A \)).

Let us consider a practical case. For an steady flow in a horizontal open channel of rectangular cross-section, and assuming a hydrostatic pressure distribution, the momentum equation (2.15) becomes:

\[ \rho Q (\beta_2 V_2 - \beta_1 V_1) = \left( \frac{1}{2} \rho g d_1^2 B_1 \right) - \left( \frac{1}{2} \rho g d_2^2 B_2 \right) \]

where \( Q \) is the discharge (m³/s), \( d \) is the flow depth, \( B \) is the channel width and \( \beta \) the momentum correction coefficient. If the shape of the velocity distribution does not change substantially between the cross-sections 1 and 2, the variations of momentum correction coefficient are negligible. The momentum equation becomes:

\[ \rho \beta Q (V_2 - V_1) = \left( \frac{1}{2} \rho g d_1^2 B_1 \right) - \left( \frac{1}{2} \rho g d_2^2 B_2 \right) \]

**The Bernoulli equation**

The ‘local form’ of the Bernoulli equation can be deduced from the Navier–Stokes equation.

Considering \( \{H1\} \) the flow is along a streamline, assuming that \( \{H2\} \) the fluid is frictionless (i.e. \( F_{\text{visc}} = 0 \)), \( \{H3\} \) the volume force potential (i.e. gravity) is independent of the time
(i.e. $\partial U/\partial t = 0$); for $\{H4\}$ a steady flow (i.e. $\partial V/\partial t = 0$) and $\{H5\}$ an incompressible flow (i.e. $\rho = \text{constant}$), the Navier–Stokes equation (2.10) along the streamline becomes:

$$\rho \nu \frac{dr}{ds} = -\rho g \frac{dz}{ds} - \frac{dP}{ds}$$  \hspace{1cm} (2.16)

where $\nu$ is the velocity along the streamline, $s$ is the direction along the streamline. A streamline is defined as an imaginary line that is everywhere tangent to the fluid velocity vector. There is no flow across a streamline and the velocity is aligned in the $s$-direction. The above equation can be re-arranged as:

$$\rho \nu \frac{dv}{d} = -\rho g dz - dP$$

and it can be rewritten as:

$$\frac{dP}{\rho} + g dz + \left(\frac{v^2}{2}\right) = 0$$  \hspace{1cm} (2.17)

The integration of equation (2.17) along a streamline yields:

$$\frac{P}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$  \hspace{1cm} (2.18)

Equation (2.18) is the Bernoulli equation. Equation (2.17) is called the differential form of the Bernoulli equation. Each term of the Bernoulli equation may be interpreted by analogy as a form of energy:

1. $P/\rho$ is analogous to the flow work per unit of mass of flowing fluid (net work done by the fluid element on its surroundings while it is flowing),
2. $U = gz$ is similar to the potential energy per unit mass,
3. $v^2/2$ is related to the kinetic energy per unit mass.

If there are no friction losses, the sum of the fluid's potential energy, kinetic energy and pressure work is a constant. Along a streamline the flow 'energy' may be re-arranged between kinetic energy (i.e. velocity), potential energy (i.e. altitude) and pressure work (i.e. flow depth) but the sum of all the terms must remain constant.

**Notes**

1. The Bernoulli equation is named after the Swiss mathematician Daniel Bernoulli (1700–1782) who developed the equation in his 'Hydrodynamica, de viribus et motibus fluidorum' textbook (first draft in 1733, first publication in 1738, Strasbourg) (Carvill, 1981; Garbrecht, 1987a: pp. 245–258).
2. Under particular conditions each of the assumptions underlying the Bernoulli equation may be abandoned:
   a. For a gas flow such that the change of pressure is only a small fraction of the absolute pressure (i.e. less than 5%), the gas may be considered incompressible. Equation (2.18) may be applied with an average density $\rho$.
   b. For unsteady flow with gradually changing conditions (i.e. slow emptying a reservoir), the Bernoulli equation may be applied without noticeable error.
2.2.4 The energy equation

The first law of thermodynamics for a system states that the net energy (e.g. heat and potential energy) supplied to the system equals the increase in energy of the system plus the energy that leaves the system as work is done:

$$\frac{dE}{dt} = \frac{\Delta Q_h}{\Delta t} - \frac{\Delta W_f}{\Delta t}$$

(2.19)

where $E$ is the total energy of the system, $\Delta Q_h$ is the heat transferred to the system and $\Delta W_f$ is the work done by the system. The energy of the system is the sum of (1) the potential energy term $gz$, (2) the kinetic energy term $v^2/2$ and (3) the internal energy $e$.

The work done by the system on its surroundings includes the work done by the pressure forces:

$$\Delta W_p = \Delta t \int_{S_1} P v dA$$

and the work done by shear forces (i.e. on a rotating shaft) is $\Delta W_s$.

For a steady and one-dimensional flow through a control volume the first law of thermodynamics becomes:

$$\frac{\Delta Q_h}{\Delta t} + \left( \frac{P_1}{\rho_1} + g z_1 + \frac{v_1^2}{2} \right) \rho_1 v_1 A_1 = \frac{\Delta W_s}{\Delta t} + \left( \frac{P_2}{\rho_2} + g z_2 + \frac{v_2^2}{2} \right) \rho_2 v_2 A_2$$

(2.20)

where the subscripts 1 and 2 refer to the upstream and downstream flow conditions, respectively.

Since the flow is steady the conservation of mass implies:

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

(2.21)

and dividing the first law of thermodynamics by $(\rho vA)$ the energy equation in differential form becomes:

$$dq_n - dw_s = d\left( \frac{P}{\rho} \right) + (g \ dz + \nu \ dv + de)$$

(2.22)

where $e$ is the internal energy per unit mass, $q_n$ is the heat added to the system per unit mass and $w_s$ is the work done (by shear forces) by the system per unit mass. For a frictionless fluid (reversible transformation) the first law of thermodynamics may be written in terms of the entropy $S$ as:

$$de = T \ dS - P \ d\left( \frac{1}{\rho} \right)$$

(2.23)
The Clausius inequality states that:

\[ dS > \frac{dq_h}{T} \]

Replacing the internal energy by the above equation and calling ‘losses’ the term \((T dS - dq_h)\), the energy equation becomes:

\[ \left( \frac{dP}{\rho} + g \, dz + v \, dv \right) + d w_s + d(\text{losses}) = 0 \]  \hspace{1cm} (2.24)

In absence of work of shear forces (i.e. \( w_s = 0 \)) this equation differs from the differentiation of the Bernoulli equation (2.17) by the loss term only.

**Notes**

1. For non-steady flows the energy equation (for two particular cases) becomes:
   - (a) for perfect and frictionless gas without heat added to the system, the energy equation is:
     \[ \rho C_p \frac{dT}{dt} = \frac{dP}{dt} \]
   - (b) for a incompressible and undilatable fluid, the energy equation is:
     \[ \rho C_p \frac{dT}{dt} = \kappa \Delta T + \Psi + \Phi \]

   where \( C_p \) is the specific heat at constant pressure, \( \kappa \) is the thermal diffusivity, \( \Psi \) is the volume density of heat added to the system and \( \Phi \) is the dissipation rate. The knowledge of the density \( \rho \) (from the continuity equation) and the pressure distribution (from the Navier–Stokes equation) enables the calculation of the temperature distribution.

2. If work is done on the fluid in the control volume (i.e. pump) the work done by shear forces \( w_s \) is negative.

### 2.3 EXERCISES

Considering a circular pipe (diameter 2.2 m), the total flow rate is 1600 kg/s. The fluid is slurry (density 1360 kg/m³). What is the mean flow velocity?

Water flows in a trapezoidal open channel (1V:3H sideslopes, 1 m bottom width) with a 1.2 m/s mean velocity and a 0.9 m water depth (on the channel centreline). (a) Calculate the mass flow rate. (b) Compute the volume discharge.

During a cyclonic event, a debris flow (density 1780 kg/m³) discharges down a trapezoidal open channel (1V:1H sideslopes, 2 m bottom width). The estimated mass flow rate is 4700 kg/s and the mean velocity is about 1.7 m/s. (a) Calculate the volume flow rate. (b) Compute the water depth on the channel centreline.

Note: For more information on debris flow, see Chapters 9 and 11.
Fundamental equations

For a two-dimensional steady incompressible flow, write the Navier–Stokes equation in Cartesian co-ordinates.

Considering a sluice gate in a horizontal smooth rectangular channel, write the momentum and Bernoulli equations as functions of the flow rate, channel width, upstream and downstream depths and the force of the gate onto the fluid only.

Considering a spherical air bubble (diameter $d_{ab}$) submerged in water with hydrostatic pressure distribution and rising freely at a constant speed $v$ (a) develop the relationship between buoyancy and forces. (b) If the drag force is expressed as:

$$\text{Drag} = C_d \frac{\rho}{2} v^2$$

where $\rho$ is the water density and $C_d$ is the drag coefficient, express the relationship between the bubble rise velocity, the fluid properties, the bubble diameter and the drag coefficient. (c) For a bubble diameter of 0.5 mm, calculate the bubble rise velocity in still water. Deduce the drag coefficient from Fig. 7.4, where the drag coefficient is plotted as a function of the particle Reynolds number: $\nu d_{ab} / \nu$. 

Summary

Applications 

3.2 Applying the Bernoulli equation in a streamline, unidimensional and incompressible fluid.

$\nu$ is the fluid density.

Reynolds number $Re = \frac{\nu d_{ab}}{\nu}$

where $\nu$ is the fluid density.